# Acoustical impedance defined by wave-function solutions of the reduced Webster equation

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The electrical impedance was first defined by Heaviside in 1884, and the analogy of the acoustical impedance was made by Webster in 1919. However, it can be shown that Webster did not draw a full analogy with the electromagnetic potential, the potential energy per unit charge. This paper shows that the analogous "acoustical potential," the potential energy per unit displacement of fluid, corresponds to the wave function $\Psi$ of the reduced Webster equation, which is of Klein-Gordon form. The wave function is found to obey all of Dirichlet, Von Neumann, and mixed (Robins) boundary conditions, and the latter give rise to resonance phenomena that are not elucidated by Webster's analysis. It is shown that the exact Heaviside analogy yields a complete analytic account of the one-dimensional input impedance, that accounts for both plane- and dispersive-wave propagation both at the origin and throughout the duct.

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## I. INTRODUCTION

Although the electric telegraph was invented by Morse in 1838, and the first commercial telegraph line was erected in 1844 between New York and Washington, much was unknown at the time about transmission line theory and the propagation of signals along wires. It was Kirchoff who introduced circuit theory in 1845, enabling the analysis of electrical transients by Kelvin (1847), Helmholz (1851), and Maxwell (1865). Nevertheless, the theory could not account for the observed decay in signal strength with distance, which limited the success of the first transatlantic cable, laid in 1858. It was eventually Heaviside who formulated the complete theory of the "distortionless line" and introduced the general concept of electrical impedance, in 1884 (see [1] for a full historical review). Although these advances were noted by Rayleigh in the second edition of *The Theory of Sound* (1894) [2], the analogy of an acoustical impedance was not drawn until 1919, by Webster [3].

The impedance has since become one of the most important quantities in acoustics. In particular, the input impedance identifies the resonances and antiresonances of ducts and pipes and can be used to reconstruct the shape of an object. In this paper, however, it is shown that the definition made by Webster is not the most complete possible, but that an exact analogy can be drawn by examining solutions of the acoustical Klein-Gordon equation.

## II. THE ACOUSTICAL IMPEDANCE

In an ac circuit driven by a seat of emf $\epsilon$, the complex electrical impedance, is

$$Z = \frac{\epsilon}{I}, \tag{1}$$

where $I$ is the alternating current. The emf $\epsilon = dW/dq$, is defined in terms of the difference in electromagnetic potential that holds between the terminals of the seat; that is, by the amount of work $dW$ done in moving unit charge $dq$ from low to high potential. The electromagnetic potential is, of course, defined as the potential energy per unit charge in order for the quantity to be independent of an arbitrary charge magnitude.

For a plane sound wave in a pipe of cross-sectional area $S$, Webster [3] adopted an electrical analogy and, likening particle displacement $\xi$ to charge $q$, defined the acoustical impedance as

$$Z = \frac{p}{u}, \tag{2}$$

with $p(x,t)$ the excess pressure and $u(x,t) = Sv(x,t)$ the "volume velocity," where $v(x,t) = \partial\xi(x,t)/\partial t$ is the particle velocity.

Since $p^2$ is proportional to the potential energy per unit volume $dV = Sd\xi$ of fluid [4]; that is, to the potential energy density, it can immediately be seen that, unlike the electrical quantities, neither the "acoustical emf" nor the impedance are stated in terms of unit "charge." This is to say that the Webster derivations do not invoke the important concept of the electromagnetic potential, which we note would have an acoustical analogy in the potential energy per unit displacement of fluid, proportional not to $p^2$ but to $p^2S$. Nevertheless, the comparison of acoustical pressure to electrical emf has led to the definition of quantities such as the specific acoustic impedance $z = SZ$, and the mechanical impedance $Z_r = zS$ [5], that have become among the most fundamental in acoustics.

In the plane-wave framework adopted by Webster, and denoting an incident wave traveling in the positive $x$ direction as $p_i = Ae^{i(\omega t - kx)}$ and the wave reflected back from a change in area at some point $x=0$ as $p_r = Be^{i(\omega t + kx)}$, the complex acoustic impedance is

$$Z = \frac{p_i + p_r}{u_i + u_r}. \tag{3}$$

For propagation along a uniform and infinitely long pipe, so that there is no reflected wave, application of the Euler equation

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$$\frac{\partial v(x,t)}{\partial t} = -\frac{1}{\rho}\frac{\partial p(x,t)}{\partial x} \tag{4}$$

yields the characteristic impedance $Z=\rho c/S$, with $\rho$ the equilibrium density of the medium and $c$ the speed of sound. More generally, for a varying cross section, it defines the "input" impedance at the origin in terms of the reflection coefficient $R=B/A$, as

$$Z_0 = \frac{\rho c}{S_0}\frac{1+R}{1-R}, \tag{5}$$

where $R$ is real and frequency independent in the case of a single step change in area, but otherwise generally complex and a function of wave number, $k=\omega/c$.

Equations (3) and (5) are among the most fundamental in duct acoustics. When analyzed with reference to Von Neumann or Dirichlet boundary conditions, the solutions give readily interpreted information about the resonance and antiresonance characteristics of a test object. For the Dirichlet condition $v(0,t)=0$ on the particle velocity, for example, corresponding to a high-impedance source (rigid termination), the resonances are identified with the poles of the magnitude of the input impedance. Conversely, for the Dirichlet condition $p(0,t)=0$ on the pressure, corresponding to an ideal, radiationless, open end, the resonances are identified with the zeroes. The Euler and continuity equations yield the corresponding Neumann conditions; namely, $[\partial p(x,t)/\partial x=0]_{x=0}$ and $[\partial v(x,t)/\partial x=0]_{x=0}$, respectively.

It may be noted, however, that the derivation of Eq. (5) assumes strictly plane-wave propagation both at the input boundary and within the duct, whereas it is known that pressure fluctuations with change in cross section are accompanied by swelling wave fronts and wave dispersions that give rise to phase velocity and resonance shifts [4,6], even in the absence of circumferential modes [7]. It can now be shown that a dispersive rather than a strictly plane-wave analysis does, in fact, lead to a definition of the impedance that is more precisely analogous to the electrical theory of Heaviside, and that the theory accounts for frequency-dependent boundary phenomena not elucidated by Webster's analysis of 1919.

It was first noted by Salmon in 1946 [8] that, for one-dimensional wave propagation in the linear, nonviscous, and adiabatic approximations, the excess pressure and area functions of the Webster equation

$$\frac{\partial^2 p(x,t)}{\partial t^2} = c^2\left\{\frac{\partial^2 p(x,t)}{\partial x^2} - \frac{1}{S(x)}\frac{dS(x)}{dx}\frac{\partial p(x,t)}{dx}\right\} \tag{6}$$

are not strictly independent variables. Whereas the potential energy per unit volume of fluid fluctuates significantly with change in cross section, Salmon noted that, in a propagating wave, the potential energy per unit length $d\xi$ must be conserved over a cycle $\tau$; that is,

$$\langle p^2(x,t)\rangle_\tau S(x) = \text{const}. \tag{7}$$

Defining a slowly varying acoustical "wave function" $\Psi(x,t)$, as

$$\Psi(x,t) = p(x,t)\sqrt{S(x)}, \tag{8}$$

thus yields the simplified or "reduced" form of the Webster equation

$$\frac{\partial^2\Psi(x,t)}{\partial t^2} = c^2\left\{\frac{\partial^2\Psi(x,t)}{\partial x^2} - U(x)\Psi(x,t)\right\}, \tag{9}$$

of Klein-Gordon form [4,9]. Equation (9) has been shown [10] to apply to one-dimensional wave propagation in general and, although the time dependencies differ, it has been noted [8,10] that the time-independent part, namely,

$$\frac{\partial^2\psi(x)}{\partial x^2} + [k^2 - U(x)]\psi(x) = 0, \tag{10}$$

for eigenfunctions $\psi(x)$, is mathematically analogous to that of the Schrödinger equation. For plane-wave propagation, an acoustical "potential function" is defined as a scaled curvature of the duct; namely, as

$$U(x) = \frac{d^2\sqrt{S(x)}/dx^2}{\sqrt{S(x)}}. \tag{11}$$

Thus, the descriptive formalism of modern wave mechanics can be applied to the macroscopic physical system.

For complex amplitude coefficients $A(k)$ and $B(k)$, and setting $\hat{k}(x)\equiv\sqrt{k^2-U(x)}$, it has been shown [11] that for $U(x)$ approximately constant over several wavelengths, such that

$$\frac{d\hat{k}(x)}{dx} \ll [\hat{k}(x)]^2, \tag{12}$$

then the harmonic solutions of Eq. (9) can be written

$$\Psi(x,t) \approx A(k)e^{i[\omega t-\hat{k}(x)x]} + B(k)e^{i[\omega t+\hat{k}(x)x]}, \tag{13}$$

or, more precisely, within the WKB approximation [12], as

$$\Psi(x,t) = A(k)e^{i[\omega t-\int_{x_0}^{x}\hat{k}(x')dx']} + B(k)e^{i[\omega t+\int_{x_0}^{x}\hat{k}(x')dx']}. \tag{14}$$

Such dispersive "wave function" solutions of the reduced Webster equation elucidate significant variations in phase velocity from predictable phenomena, and previous work [4,6] has considered in detail those due to piecewise constant potential functions, for which $U(x)=U_0$.

It is now possible to present a wave-mechanical account of the acoustical impedance. We have previously noted that the Webster definition did not invoke the important concept of the electromagnetic potential, which would have an acoustical analogue in the potential energy per unit displacement of fluid. This "acoustical potential" [quite distinct from the potential function $U(x)$] would be proportional not to $p^2$, but to $p^2S$. Given the insight of Salmon [Eq. (7)], it is immediately obvious that such a quantity corresponds to the squared wave function $\Psi^2$. The acoustic impedance in the more exact Heaviside analogy is, thus, immediately defined as

$$Z_H = \frac{\Psi}{u}, \tag{15}$$

so that $Z=Z_H/\sqrt{S}$ [cf. (2)]. The implications for duct resonance can now be examined.

### III. THE HEAVISIDE INPUT IMPEDANCE

We begin by noting that the general harmonic solutions at the origin are given in terms of wave functions

$$\Psi(0,t) = A(k)[1 + R(k)]e^{i\omega t}, \tag{16}$$

where $R(k)=B(k)/A(k)$ is the reflection coefficient due to an arbitrary sequence of potential functions (the absence of bound states is assumed), and may include the effects of a radiation impedance [4]. The amplitude $A(k)$ may be found as follows. Since from (8) it is the case that

$$\frac{\partial p(x,t)}{\partial x} = \frac{1}{\sqrt{S(x)}}\frac{\partial \Psi(x,t)}{\partial x} - \frac{1}{2[S(x)]^{3/2}}\frac{dS(x)}{dx}\Psi(x,t), \tag{17}$$

the Euler equation (4) states that

$$u(x,t) = \frac{1}{\rho}\sqrt{S(x)}\int \frac{1}{2S(x)}\frac{dS(x)}{dx}\Psi(x,t) - \frac{\partial \Psi(x,t)}{\partial x}dt. \tag{18}$$

The assumption of a piecewise constant or slowly varying potential function at the origin allows the substitution of Eq. (14) [or (13)] into (18). The coefficient $A(k)$ can then be found by setting the condition of a high-impedance harmonic source, for which

$$u(0,t) = e^{i\omega t}. \tag{19}$$

Evaluation of Eq. (18) at the origin then yields

$$A(k) = \frac{\rho\omega}{\sqrt{S_0}}\frac{1}{\hat{k}_0}\left(\frac{1}{1 - R(k) - \dfrac{i\alpha[1 + R(k)]}{\hat{k}_0}}\right), \tag{20}$$

where

$$\hat{k}_0 \equiv \hat{k}(0) \equiv \sqrt{k^2 - U(0)} \tag{21}$$

and

$$\alpha = \left.\frac{1}{2S_0}\frac{dS(x)}{dx}\right|_{x=0}. \tag{22}$$

The "Heaviside" input impedance is finally obtained from Eqs. (15), (16), and (19), as

$$Z_{H_0} = \frac{\rho\omega}{\sqrt{S_0}}\frac{1}{\hat{k}_0}\frac{1 + R(k)}{1 - R(k) - \dfrac{i\alpha[1 + R(k)]}{\hat{k}_0}}, \tag{23}$$

and is directly proportional to the measurable pressure [cf. (8)], so that $Z_0=Z_{H_0}/\sqrt{S_0}$.

Recent work [13,14] has suggested that the condition of a high-impedance source can be met for experimental measurements up to around 5 kHz. At higher frequencies, separate measurements of velocity [15,16] may be required and calibrated terms should be introduced into Eq. (20).

Although the effects on the impedance are complicated when there is both a dispersion and a gradient at the input, two special cases may usefully be examined.

(1) $\alpha=0$

If the initial gradient is zero, $dS(x)/dx=0$ at $x=0$, then

$$A(k) = \frac{\rho\omega}{\sqrt{S_0}}\frac{1}{\hat{k}_0}\left(\frac{1}{1 - R(k)}\right), \tag{24}$$

and Eq. (23) reduces to

$$Z_{H_0} = \frac{\rho\omega}{\sqrt{S_0}}\frac{1}{\hat{k}_0}\frac{1 + R(k)}{1 - R(k)}. \tag{25}$$

For $U(0)=U_0$, the term $\hat{k}_0$ describes the effects of a dispersion due to a section of catenoidal ($U_0>0$) or cosinusoidal ($U_0<0$) horn [4], and for known $\rho$, $\omega$, and $S_0$, the value of $U_0$ may be recovered in the high-frequency limit, as $R(k)\to 0$.

(2) $\alpha\neq 0$, $\hat{k}_0=k$

Since $U(0)=0$, the term $\alpha$ describes the initial gradient or angle of a conical duct, with linear radius function $r(x)$:

$$r(x) = r(0)(1 + \alpha x). \tag{26}$$

The input impedance

$$Z_0 = \frac{\rho c}{S_0}\frac{1 + R(k)}{1 - R(k) - \dfrac{i\alpha[1 + R(k)]}{k}} \tag{27}$$

demonstrates frequency-dependent departures from the plane-wave solution (5).

It may be noted that there are no corrections for the ideal radiationless case $R=-1$, since $\Psi(0,t)=0$ in Eq. (17), and the boundary condition is purely Dirichlet. In fact, Eq. (23) states that for an ideal open input, that is, at a common pressure node, neither wave dispersion nor duct gradient affect the impedance which remains as $Z=0$, despite the singularity in the plane-wave potential function [10]. A similiar phenomenon has previously been observed for eigenvalue perturbations at nodes [6].

In the context of the inverse problem of reconstructing an unknown potential function $U(x)$ from acoustic measurements, Aktosun [17] has indicated that $\alpha$ can be identified in the high-frequency limit, as $\alpha=i\lim_{k\to\infty}\{k[\Lambda(k)+1]\}$, where $\Lambda(k)=Z_0S_0/(\rho c)$. When $\alpha$ is known, $R(k)$ is determined as

$$R(k) = \frac{\Lambda(k)(1 - i\alpha/k) - 1}{\Lambda(k)(1 + i\alpha/k) + 1}. \tag{28}$$

From $R(k)$, a unique potential function can be reconstructed according to standard methods [18,19]. Further, when $S(0)$ and $[dS(x)/dx]_{x=0}$ are individually specified, the area function can also be obtained. Thus, the results presented here are essential in the noninvasive measurement of acoustical ducts, such as the vocal tract, that cannot be assumed uniform at the source of excitation. Nevertheless, the accuracy of the formulae will be limited in regions of rapid flare (of more significance in wide-bore musical and engineering acoustics applications) due to the one-dimensional approximation and the absence of terms accounting for higher-order duct modes.

The foregoing analysis constitutes a full analogy with the electrical transmission line theory of Heaviside. It is particularly remarkable that the analysis invokes a mixed (Robins) boundary condition on the wave function, of the form

$$\left.\frac{\partial\Psi(x,t)}{\partial x}\right|_{x=0} - \alpha\Psi(0,t) = 0, \tag{29}$$

whereas only Dirichlet or Neumann conditions can usually be set on the excess pressure (or velocity). Further, the departure from purely Neumann conditions is described by the constant $\alpha$, and it is notable that any set of area functions sharing a value of $\alpha$, and corresponding to a single potential function, will have a completely identical spectrum, up to a constant amplitude factor. This finding extends previous accounts of the phenomenon of nonunique "many-to-one mappings" between the shape of an object and its peak resonance frequencies (cf. [20]).

## IV. CONCLUSIONS

It is widely considered that the definition of the acoustical impedance, first made by Webster in 1919 and having since stood in the literature, exists in precise analogy with the electrical impedance, originally introduced by Heaviside in 1884. In particular, the excess pressure is compared to the electrical emf. However, this paper points out that the exact analogy is made by defining an acoustical potential, as the potential energy per unit displacement of fluid. It is shown that such a quantity corresponds not to the pressure, but to the wave function $\Psi$ of the reduced Webster equation. Since the wave function obeys mixed as well as Dirichlet and Neumann boundary conditions, wave function solutions of the Webster equation are found to yield a full and complete analysis of the one-dimensional input impedance that accounts for frequency-dependent boundary phenomena not elucidated by standard analyses.

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